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Camassa–Holm equation: transformation to deformed sinh–Gordon equations, cuspon and soliton solutions

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Abstract. In this paper a relation between the Camassa–Holm equation and the non-local deformations of the sinh–Gordon equation is used to study some properties of the former equation. We will show that cuspon and soliton solutions can be obtained from soliton solutions of the deformed sinh–Gordon equation.

1. Introduction

The Camassa–Holm equation (CH)

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

appeared first in a physical context describing the shallow-water approximation in inviscous hydrodynamics [1]. The variable $u(x,t)$ represents the fluid velocity in the horizontal direction x, and κ is a constant. Although first derived by Hamiltonian approximation methods, it can also be obtained by using standard asymptotic methods [2], in the same way one obtains the Korteweg–de Vries (KdV) or the Benjamin–Bona–Mahoney (BBM) equations [3]. Still from the physical point of view it can be considered as a higher-order nonlinear generalization of the BBM equation, which is obtained when the right-hand side of equation (1) is dropped. Much of the interest in this equation comes from two remarkable facts: (a) it is a completely integrable equation [1, 4], consequently allowing the use of many peculiar properties of these systems [5, 6]; (b) it possesses peaked solitary-wave solutions (termed peakons) in the limit $\kappa \to 0$. Peakons are solutions presenting a finite discontinuity in its first derivative, such as the solution found in [1]:

$$u(x,t) = c\exp(-|x - ct|) \tag{2}$$

where c is an arbitrary constant. Further studies led to new solutions such as the billiard solutions [7] and the cuspon solutions [8–10] (solutions with the first derivative going to infinity at a given point [11]). Also the term with κ can be set to zero in equation (1) through the map $u \to u - \kappa$, $x \to x + \kappa t$. This equation with $\kappa \neq 0$ has essentially different classes of solutions, which vanish at infinity than with $\kappa = 0$: peakons if $\kappa = 0$ and cuspons otherwise.

In the derivation of the CH equation in shallow-water theory, the constant κ is given in terms of the physical variables g and h_0, the acceleration of gravity and the undisturbed depth by $\kappa^2 = gh_0$. It is clear that $\kappa = 0$ is a non-physical case, at least in normal gravity conditions (under microgravity conditions surface-tension effects have to be taken into account, and this

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has not be done in the present case). It is thus of general interest to study the solutions of the CH equation with $\kappa \neq 0$.

In this paper we concentrate on the investigation of the CH equation by making use of its relation with the deformed sinh–Gordon equation (see below) [12]. This equation is a completely integrable one, and a transformation between these equations allows us to produce new solutions for one of them, given the solutions of the other. This relation is based on the hodograph transformation and becomes evident if the $\bar\partial$-dressing [13–15] procedure for the CH is taken into the consideration [10]. This becomes possible due to the modification of the dressing procedure given in [16, 17]. We will show that this relation is described in terms of the solutions of an ordinary differential equation which, in particular cases, reduces to a (transcendental) algebraic equation. We will first give a purely algebraic approach to this transformation, not involving the $\bar\partial$-problem in section 2. Next, using results on the relation between the $\bar\partial$-problems for CH and the deformed sinh–Gordon equations given in the appendix we will investigate the solutions of the CH equation originating from the one-soliton solution of the deformed sinh–Gordon equation. The general feature here is that each solution is mapped into a family of solutions of the CH equation. Elements of this family involve solutions with a local real extremum. This extremum can be of the cuspon type or of the soliton type, i.e. the space derivatives of these solutions tend to infinity or equal zero in the extremal points. This will be discussed in section 3. As a next step, we will analyse some interesting examples, when the local cuspon and soliton solutions become global solutions, i.e. determined and bounded for all values of the independent parameters x and t, but we will show that this happens *either* for the soliton solution or for the cuspon solution in given family. We will give examples of explicit one-cuspon and one-soliton solutions and will comment on multi-soliton (cuspon) solutions. The paper closes with an appendix consisting of a series of results on $\bar\partial$-problems for CH and the deformed sinh–Gordon equations.

2. Transformation between CH and the deformed sinh–Gordon equation

In what follows it will be more convenient to work with the CH equation written in the following form:

$$u_{xxt} - a^2 u_t + 4u_x - uu_{xxx} + 3a^2 uu_x - 2u_x u_{xx} = 0 \qquad a = \text{constant} \tag{3}$$

which is a transformed from of equation (1). To be explicit, the transformation of equation (3) to (1) is given by the following:

$$a \to 1 \qquad u \to \frac{2}{\kappa}u \qquad t \to -\frac{\kappa}{2}t$$

for the case $\kappa \neq 0$ which is considered here.

We will show that there exists a transformation relating equation (3) to the deformed sinh–Gordon equation given by

$$\tfrac{1}{2}\chi_{TX} - c_1 \mathrm{e}^{\chi} - c_2 \mathrm{e}^{-\chi} - c_3\big(\partial_X^{-1}\big(\mathrm{e}^{\chi}\big)\partial_X^{-1}\big(\mathrm{e}^{-\chi}\big)\big)_X = 0 \tag{4}$$

where c_k $(k = 1, 2, 3)$ are arbitrary constants.

As both equations are integrable by the inverse scattering transform, they can be given as solvability conditions for linear overdetermined systems of equations. We will give the transformation between equations (3) and (4) by relating their associated linear systems.

The linear overdetermined system for equation (4) has the form [12]

$$\Psi_{aXX} - U_2\Psi_{aX} - \frac{1}{\Omega}\Psi_a = 0 \tag{5}$$

$$\Psi_{aT} + V\Omega\Psi_{aX} + W\Psi_a = 0 \tag{6}$$

where Ω is a constant parameter. It easy to check that the compatibility condition of this system is the nonlinear system of equations for the variables U_2, V and W,

$$2W_X + U_{2T} = 0 \tag{7}$$

$$(V_X + VU_2)_X = 0 \tag{8}$$

$$W_{XX} - U_2 W_X + 2V_X = 0. \tag{9}$$

This system can be easily reduced to equation (4) by taking

$$\chi_X = U_2 \tag{10}$$

and integrating equation (7)–(9) with respect to the X variable, thus obtaining

$$W = -\tfrac{1}{2}\chi_T + c(T) \tag{11}$$

$$V = c_3(T)\partial_X^{-1}\left(\mathrm{e}^{\chi}\right)\mathrm{e}^{-\chi} + c_2(T)\,\mathrm{e}^{-\chi} \tag{12}$$

$$W_X = -2\partial_X^{-1}\left(V_X \mathrm{e}^{-\chi}\right)\mathrm{e}^{\chi} - c_1 \mathrm{e}^{\chi} \tag{13}$$

where equation (11) has already been used in equations (12) and (13). Substituting equation (12) into (13) and using equation (11), gives (4) when c_k $(k = 1, 2, 3)$ are constants.

To go over to the linear problem for the CH let us introduce a hodograph transformation by means of the new independent variables (x, t) through

$$X = \Phi(x, t) \qquad T = t \tag{14}$$

where Φ is an arbitrary function of its arguments. In these variables the system (5) and (6) takes the form

$$\Psi_{axx} - \tilde{U}_2\Psi_{ax} - \frac{\tilde{U}_1}{\Omega}\Psi_a = 0 \tag{15}$$

$$\Psi_{at} - (u - \tilde{V}\Omega)\Psi_{ax} + \tilde{W}\Psi_a = 0 \tag{16}$$

where

$$\begin{aligned} &u = \frac{\Phi_t}{\Phi_x} \qquad \tilde{V} = V(\Phi_x)^{-1} \qquad \tilde{W} = W \\ &\tilde{U}_1 = (\Phi_x)^2 \qquad \tilde{U}_2 = U_2\Phi_x + \frac{\Phi_{xx}}{\Phi_x}. \end{aligned} \tag{17}$$

Now, as the linear system given by equations (5) and (6) implies a compatibility condition, a corresponding compatibility condition for the transformed system will emerge, which is given by

$$\tilde{V}_{xx} + (\tilde{U}_2\tilde{V})_x = 0 \tag{18}$$

$$-u_{xx} + 2\tilde{W}_x + \tilde{U}_{2t} - (\tilde{U}_2 u)_x = 0 \tag{19}$$

$$\tilde{W}_{xx} - \tilde{U}_2\tilde{W}_x + \tilde{V}\tilde{U}_{1x} + 2\tilde{V}_x\tilde{U}_1 = 0 \tag{20}$$

$$\tilde{U}_{1t} - u\tilde{U}_{1x} - 2u_x\tilde{U}_1 = 0. \tag{21}$$

Alternatively, this equation could be obtained by substitution of the hodograph transformation (14) directly into equations (7)–(9). We notice that in view of equations (17), the last equation above, equation (21) is an identity. However, precisely *this* equation will play an important role in the derivation of CH.

Because there are more potentials in equations (18)–(21) than the number of equations, any *solvable* reduction can be put on this system. We consider this new reduction as equation defining the function Φ (which remained arbitrary up to now). The simplest equation

$$\tilde{V} = 1 \tag{22}$$

leads to the CH, equation (3) for the potential u simply after substitution of $\tilde{U}_1$ from equations (18)–(20) into (21). In view of (22), the overdetermined linear system (15) and (16) takes the form

$$\Psi_{axx} - a\Psi_{ax} + \frac{m-2}{2\Omega}\Psi_a = 0 \qquad m = u_{xx} - a^2 u \tag{23}$$

$$\Psi_{at} - (u - \Omega)\Psi_{ax} + \tfrac{1}{2}(u_x + au)\Psi_a = 0. \tag{24}$$

This form of the linear system is more suitable for our consideration and can be reduced to that introduced in [1] by a simple transformation, $\Psi_a \to \Psi_a \exp(ax/2 - a\Omega t/2)$.

Note that equation (22) is an algebraic one for the function Φ, due to the hodograph transformation (14), once the function $V(X, T)$ is known. However, the relation between function V and the solution χ is a differential one. To clarify the structure of equation (22) and its relation with the solution χ of equation (4), let us integrate equation (18) by taking into account equation (22). One obtains that $\tilde{U}_2$ is an arbitrary function of t. We take this function to be constant to obtain equation (3) with the constant coefficients

$$\tilde{U}_2 = a = \text{constant}.$$

Then, the first and the last equations of the system (17) give the differential relation between the functions χ and u:

$$u = \frac{\Phi_t}{\Phi_x} \tag{25}$$

$$a = U_2\Phi_x + \frac{\Phi_{xx}}{\Phi_x}. \tag{26}$$

Note that the function U_2 in these equations depends on (x, t) due to equation (14), but the relation with χ is given by equation (10) in terms of the variables (X, T). We can integrate equation (26) in x, obtaining

$$\ln(\Phi_x) + \chi = ax + d(t) \tag{27}$$

where d is an arbitrary function of t, and χ is a function of $X = \Phi(x, t)$ and $T = t$. We thus have an *ordinary* differential equation for the unknown function Φ, which is related to the solutions of CH by equation (25). The time dependence of Φ is determined by the time dependence of the solution χ.

Thus we have shown that any given solution of the deformed sinh–Gordon equation (4) gives rise to a *family* of solutions of CH equations through relations (14), (25) and (27). However, the reverse procedure (obtaining the solutions of the deformed sinh–Gordon equation through the CH equation) is much more complicated, because one needs to find the function Φ through integration of equation (25).

However, it will turn out that many interesting solutions of CH, like the ones considered in section 3, do not originate from solitonic solutions of equation (4). At this point we have to resort to a transformation operating on equation (4), in fact, a Miura-type transformation, which takes equation (4) to an equivalent one. This is done by eliminating the parameter Ω from equations (5) and (6) and by using the gauge transformation

$$\Psi_a = H\Psi \qquad 2\frac{H_x}{H} - U_2 = -a.$$

The system (5) and (6) results in a system for Ψ given by

$$\Psi_{XT} - A\Psi_T - v\Psi = 0 \tag{28}$$

$$\Psi_{XX} - a\Psi_X - 2\partial_T^{-1} v_X \Psi - \frac{1}{\Omega}\Psi = 0 \tag{29}$$

with

$$A = \frac{U_2}{2} + \frac{V_X}{V} + \frac{a}{2} \tag{30}$$

$$v_X = \tfrac{1}{4}\left(\tfrac{1}{2}\chi_X^2 - \chi_{XX}\right)_T. \tag{31}$$

The compatibility condition for this system is

$$(v_X - 2Av + av)_X = 0 \qquad A_X + A^2 - aA - 2\partial_T^{-1}u_X = 0. \tag{32}$$

Defining now

$$\xi = \ln(v) \tag{33}$$

we obtain the new version of the deformed sinh–Gordon equation:

$$\tfrac{1}{2}\xi_{XT} - C_1 \mathrm{e}^{\xi} - C_2 \mathrm{e}^{-\xi} + C_3 \mathrm{e}^{-\xi}\partial_X^{-1}\left(\mathrm{e}^{-\xi}\right)_T = 0 \tag{34}$$

where C_k are constants, which appear after integrating the system (32).

This equation is also an integrable one. Soliton solutions for equation (34) will give rise to non-soliton solutions of equation (4), but exactly those that generate the cuspon solutions of CH. One could, of course, have discarded equation (4) from the beginning, but this would be, at least, counter-intuitive, as the relation between CH and equation (4) is quite a direct one. In order, however, to have a better understanding of the interplay between the two versions of the deformed sinh–Gordon and the CH equation, we will turn to the associated $\bar{\partial}$-problems [10, 12]. An adaptation of these methods will also give the $\bar{\partial}$-problem associated with equation (4). This last point, together with details of the $\bar{\partial}$-problems associated with CH and equation (34) are collected in an appendix.

3. Solutions of CH obtained from the soliton solutions of the deformed sinh–Gordon equation

3.1. General results

The results of the previous section show that every solution of the sinh–Gordon-type equation (4) gives rise to a family of solutions of the CH equation, (3), due to the relations (14), (25) and (27). The number of solutions in each of these families equals the number of independent solutions of equation (27). In a general situation, this last equation cannot be solved analytically. Further, for each family, the question about real solutions should be studied separately. It turns out that many simplifications arise when we consider solutions obtained through the $\bar{\partial}$-problem for equation (34). In this case, equation (27) can be replaced by an *algebraic* one, namely equation (A26). More details of this approach are given in the appendix. In what follows, we will refer to results obtained there.

Let us consider the class of the solitary wave solutions of equation (3) which is related to soliton solutions of equation (34). We will give some general results before turning to specific examples.

First of all, we will consider the one-soliton solution of equation (34). Even in this simple case, a large variety of solutions of equation (3) can be obtained, among which are the cuspons and ordinary solitons. Other solutions are not bounded at infinity. The existence of such a variety of solutions is related to the fact that equation (A26) defines both the form of the solution and the number of independent solutions in each family. This is closely related to the arbitrary constants b, p in equation (A35). A further simplification sets in here. The algebraic equation (A26) is now, in the one-soliton case, of the form (see equations (A37) and (A39))

$$ac(\gamma-1)^2(a-2a\gamma-p\chi)\chi^{1/(2\gamma-1)}\mathrm{e}^{a\eta}-p\gamma^2\chi-a(2\gamma-1)(\gamma-1)^2=0 \tag{35}$$

$$\chi=\exp\left[a(2\gamma-1)\Phi+\frac{2\gamma-1}{a\gamma(\gamma-1)}t\right] \tag{36}$$

where $\gamma=b/a$, $\beta=1/a^2\gamma(1-\gamma)$ and $\eta=x+\beta t$. The solution of CH equation u is related to χ by the equation

$$u=\beta\left(1+a(2\gamma-1)\frac{\chi}{\chi_\eta}\right) \tag{37}$$

which follows from equations (25) and (35). In the following considerations, we assume that $\gamma\neq 1$ or 0 (otherwise an irregularity appears in equations (35) and (36)) and $\gamma\neq\frac{1}{2}$ (otherwise the t dependence disappears from the solution u).

In view of (35), the solution u given by equation (37), can be represented in the following form:

$$u=\frac{(1-2\gamma)^3p\chi}{a\gamma(\gamma-1)(2\gamma a-a+p\chi)(-a+4a\gamma-5a\gamma^2+2a\gamma^3+\gamma^2p\chi)} \tag{38}$$

which will be more convenient for our investigations. It is useful to keep in mind that the map $\chi\leftrightarrow\eta$ is not reciprocal, so that χ in the right-hand side of equation (38) represents the family of solutions of the algebraic equation (35) related to the given values of the constants γ and p.

We now proceed to the investigation of the η dependence of the solution u implicitly by using equations (35) and (38), and keeping in mind that we are interested in the real solutions of CH. Note that we cannot guarantee the existence of the real solutions on the whole η-axis, $-\infty<\eta<+\infty$ (we call these solutions *global* ones). Moreover, the real solutions, localized in the neighbourhood of a given point η_k do not necessary belong to *any* fixed global solution of CH. So, the conclusions to be drawn are correct, generally speaking, only locally around the given point η_k. Some simple examples of the global bounded real solutions will be given below.

First of all, note that equation (38) implies that u can be equal to zero only at $\chi=0$ or $\chi\to\infty$. From equation (35) it follows that $\eta\to\pm\infty$ as $\chi\to 0,\infty$. This means that any real solution u (no matter whether it is a local or global one) of equations (3) has the same sign for all of the region where this solution is defined ($(-\infty,+\infty)$ for the global solution). This fact is of major importance in the construction of global solutions.

Let us obtain the derivative u_η by differentiating equation (38) and taking into account equation (35):

$$u_\eta=\frac{(2\gamma-1)^4p\chi(a-3a\gamma+2a\gamma^2+p\gamma\chi)}{\gamma(\gamma-1)(2a\gamma-a+p\chi)(a-3a\gamma+2a\gamma^2-\gamma p\chi)}\times(-a+4a\gamma-5a\gamma^2+2a\gamma^3+\gamma^2p\chi)^{-1}. \tag{39}$$

Now consider the points χ_k for which this derivative equals zero or tends to infinity. In this way, we define the points where the derivative changes sign (i.e. the extremal points). Of course, these points are not necessarily situated on the same curve χ, which is a fixed global solution of equation (3). Generally speaking, they are actually not situated on the global solutions at all.

From this last equation, we can see that the derivative u_η equals zero at three points: $\chi_1\to\infty$, χ_2, χ_3:

$$\chi_1\to\infty\qquad\chi_2=0\qquad\chi_3=\frac{a}{\gamma p}(-1+3\gamma-2\gamma^2). \tag{40}$$

In three other points, it tends to infinity:

$$\chi_4 = \frac{a}{p}(1-2\gamma) \qquad \chi_5 = \frac{a}{\gamma p}(1-3\gamma+2\gamma^2) = -\chi_3 \tag{41}$$

$$\chi_6 = \frac{a}{\gamma^2 p}(1-2\gamma)(\gamma-1)^2. \tag{42}$$

Two points, χ_3 and χ_5, have a significant meaning, because they correspond to soliton and cuspon solutions, respectively. The other points describe the asymptotic behaviour of solutions at $\eta \to \pm\infty$. This will become clear below. The soliton and cuspon are localized around the points η_3 and η_5 ($\Gamma = 1/(1-2\gamma)$)

$$\exp(a\eta_3) = \frac{\gamma}{ac(\gamma-1)}\left(\frac{a}{\gamma p}((2\gamma-1)(1-\gamma))\right)^{\Gamma} \tag{43}$$

$$\exp(a\eta_5) = (-1)^{\Gamma+1}\exp(a\eta_3). \tag{44}$$

To see this, note that around these points the solution u is represented by series, which can be constructed by expanding equations (35) and (38) in the neighbourhood of the corresponding points η_k, χ_k ($u_k \equiv u|_{\eta\to\eta_k}$)

$$u_3 = \frac{(2\gamma-1)^2}{a^2\gamma(1-\gamma)} - \frac{(2\gamma-1)^4}{16\gamma^2(\gamma-1)^2}(\eta-\eta_3)^2 + \cdots \tag{45}$$

$$u_5 = \frac{1}{a^2\gamma(1-\gamma)} + \left(\frac{3}{a^2\gamma(\gamma-1)}\right)^{2/3}(\eta-\eta_5)^{2/3} + \cdots. \tag{46}$$

From equation (44) it follows that

(a) $u_\eta|_{\eta\to\eta_5} \sim 1/(\eta-\eta_5)^{1/3}$ and

$$u_\eta \to \begin{cases} +\infty & \text{as} \quad \eta\to\eta_5+0 \\ -\infty & \text{as} \quad \eta\to\eta_5-0. \end{cases}$$

This corresponds to what is defined as a cuspon, that is, a solution with the first derivative going to infinity at a given point.

(b) Cuspon and soliton solutions always move in opposite directions, due to the definition of the velocity β. The cuspon amplitude is equal to its velocity.

Let us now write down the asymptotics at points χ_1, χ_2, χ_4 and χ_6. This will enhance our understanding of the meaning of these points, and further will take us to some results concerning global real solutions. We have

$$\exp(a\eta_1) \to -\frac{\gamma^2}{ac(\gamma-1)^2}\chi^{\Gamma} \qquad |\exp(a\eta_1)| \to \begin{cases} 0 & \text{if} \quad 2\gamma > 1 \\ \infty & \text{if} \quad 2\gamma < 1 \end{cases} \tag{47}$$

$$\exp(a\eta_2) \to -\frac{1}{ac}\chi^{\Gamma} \qquad |\exp(a\eta_2)| \to \begin{cases} 0 & \text{if} \quad 2\gamma < 1 \\ \infty & \text{if} \quad 2\gamma > 1 \end{cases} \tag{48}$$

$$\exp(a\eta_4) \to \frac{(2\gamma-1)^2}{pc(\gamma-1)^2}\left(\frac{a}{p}(1-2\gamma)\right)^{\Gamma}(\chi-\chi_4)^{-1} \qquad |\exp(a\eta_4)| \to \infty \tag{49}$$

$$\exp(a\eta_6) \to -\frac{p\gamma^4}{ca(2\gamma-1)^2(\gamma-1)^2}\left(\frac{a}{p\gamma^2}(1-2\gamma)(\gamma-1)^2\right)^{\Gamma}(\chi-\chi_6) \qquad |\exp(a\eta_6)| \to 0 \tag{50}$$

and

$$u_1 \to \frac{(1-2\gamma)^3}{a\gamma^3 p(\gamma-1)}\left(-ac\frac{(\gamma-1)^2}{\gamma^2}\exp(a\eta)\right)^{-1/\Gamma} \qquad a\eta \to \begin{cases} -\infty & \text{if} \quad 2\gamma > 1 \\ +\infty & \text{if} \quad 2\gamma < 1 \end{cases} \tag{51}$$

$$u_2 \to \frac{p(1-2\gamma)}{a^3\gamma(\gamma-1)^3}(-ac\exp(a\eta))^{1/\Gamma} \qquad a\eta \to \begin{cases} -\infty & \text{if} \quad 2\gamma < 1 \\ +\infty & \text{if} \quad 2\gamma > 1 \end{cases} \tag{52}$$

$$u_4 \to \frac{c}{a\gamma}(1-\gamma)\left(\frac{a}{p}(1-2\gamma)\right)^{-\Gamma}\exp(a\eta) \qquad a\eta \to +\infty \tag{53}$$

$$u_6 \to \frac{\gamma}{ca^3(1-\gamma)}\left(\frac{a}{\gamma^2 p}(1-2\gamma)(\gamma-1)^2\right)^{\Gamma}\exp(-a\eta) \qquad a\eta \to -\infty. \tag{54}$$

So u decreases or increases exponentially when $\eta \to \pm\infty$.

We now formulate some conclusions, which can be drawn from equations (40)–(54). These conclusions (if not specifically stated) are valid for both local and global solutions. It should be kept in mind that they concern only solutions of CH obtainable from one-soliton solutions of equation (34).

(a) From expressions (47)–(54) it follows that a real solution u may tend to infinity only at $\eta \to \pm\infty$ according to (53) and (54). They also can vanish exponentially at infinity according to (51) and (52).

(b) Any given real solution can have no more then one extremum either at the point η_3 (ordinary soliton) or at η_5 (cuspon). In fact, in order have both extrema we would need $\eta_3 = \eta_5$ due to equation (44). However, $u_3(\eta_3) \neq u_5(\eta_5)$ in view of (45) and (46).

(c) From the expansions (45) and (46) it follows that the function u, which belongs to the given family of solutions, has maximum (or minimum) at the points η_3 (or η_5).

(d) From equation (38) it follows that u can be equal to zero only at the points $\eta \pm \infty$. This means that any real solution cannot change sign over the entire region of its definition ($\infty < \eta < +\infty$, for global solutions).

(e) As far as the solution cannot change sign, the ordinary *global* soliton and cuspon vanish exponentially as $|\eta| \to \infty$ in accordance with (51) and (52) if $u_3(\eta_3) > 0$ (or $\gamma(1-\gamma) > 0$, positive amplitude) and $u_5(\eta_5) < 0$ (or $\gamma(1-\gamma) < 0$, negative amplitude), respectively. Otherwise these solutions tend to infinity (also exponentially) according to equations (53) and (54) as $|\eta| \to \infty$. The conditions $u_3(\eta_3) > 0$ and $u_5(\eta_5) < 0$ are the first necessary conditions for the existence of bounded global solitons and cuspons. It follows that the global soliton and global cuspon cannot be found in the same family of CH solutions, characterized by the given value of the real parameter γ. However, local soliton and cuspon solutions can both be found in the same family.

(f) The second necessary local condition for the existence of global real cuspon or soliton solutions, corresponding to the given constant γ, is given by equations (43) and (44). Namely, the right-hand side of these equation must be positive or equal to zero. This condition is not such a strong one, as it can always be satisfied for a cuspon or soliton (but not for both of them simultaneously) by the appropriate chose of the constants p and c in these formulae. One should remember that both of these local conditions do not guarantee the existence of global cuspons and solitons in the given family of solutions.

3.2. Examples

It is clear from equation (35) that the factor $\gamma = b/a$ determines the general structure of this equation. As has been pointed out above, there are simple examples when equation (35) is exactly solvable without numerical calculations. We will describe two such cases. In the first one, equation (35) leads to the equation of the third power (on the functions χ (ordinary soliton) or ϕ: $\chi = \phi^2$ (cuspon)). In the second case, equation (35) is of the fourth power in the functions χ or ϕ; $\chi = \phi^3$. Neither a global soliton nor a cuspon exist in this case, because this equation does not possess real solutions. Only the necessary local conditions for soliton $\left(\gamma = \frac{1}{3}, \frac{2}{3}\right)$ and cuspon ($\gamma = -1, 2$) existence are fulfilled (see the previous paragraph).

Accordingly, we will consider only the first case. Both the ordinary soliton solution and cuspon solution exist in this situation. The ordinary soliton corresponds to the factor $\gamma = \frac{3}{4}$ or $\frac{1}{4}$, and is described by equation (37)

$$u_{sol\pm} = \frac{16}{3a^2}\left(1 \mp \frac{a}{2}\frac{\chi_\pm}{\chi_{\pm\eta}}\right) \tag{55}$$

with

$$\chi_\pm = \frac{2\cos(h_\pm - \pi/3)}{\cos(3h_\pm)} - 1 \qquad h_\pm = \tfrac{1}{3}\arctan(\exp(\pm a\eta/2)) \qquad \eta = x + \frac{16}{3a^2}t.$$

The upper and lower signs '+' and '−' correspond to $\gamma = \frac{1}{4}$ and $\gamma = \frac{3}{4}$, respectively. There would, in principle, exist two soliton solutions but it can be easily shown that

$$\begin{aligned} u_{sol+} &\equiv u_{sol-} \equiv u_{sol} \\ &= \frac{16}{3a^2}\left(1 - \frac{3(\sqrt{3} + 2\sin(2h))}{(1 + 2\cos(2h))(2\sqrt{3}\cos(2h) - \sqrt{3}\cos(4h) + 2\sin(2h) + \sin(4h))}\right) \end{aligned} \tag{56}$$

$h \equiv h_+$.

For the cuspon solution ($\gamma = -\frac{1}{2}, \frac{3}{2}$) we represent equation (37) in a slightly different form in accordance with (36)

$$u_{cusp\pm} = -\frac{4}{3a^2}\left(1 \mp a\frac{\phi_\pm}{\phi_{\pm\eta}}\right) \qquad \chi_\pm = \phi_\pm^2 \tag{57}$$

where

$$\begin{aligned} &\phi_\pm = z_\pm + ((z_\pm - 1)(z_\pm + 1)^2)^{1/3} + ((z_\pm - 1)^2(z_\pm + 1))^{1/3} \\ &z_\pm = \exp(\pm a\eta) \qquad \eta = x - \frac{4}{3a^2}t \end{aligned} \tag{58}$$

the upper and lower signs '+' and '−' correspond to $\gamma = -\frac{1}{2}$ and $\gamma = \frac{3}{2}$, respectively. As well as in the soliton case, one can prove [10]) that

$$\begin{aligned} u_{cusp+} &\equiv u_{cusp-} \equiv u_{cusp} \\ &= -\frac{16}{3a^2}\frac{3e^{\eta/2}\left((-\sinh(\eta/2)\cosh(\eta/2)^2)^{1/3} + (\sinh(\eta/2)^2\cosh(\eta/2))^{1/3}\right) + 1}{12e^{-\eta}\left(-(\sinh(\eta/2)\cosh(\eta/2)^2)^{1/3} + (\sinh(\eta/2)^2\cosh(\eta/2))^{1/3} + e^{-\eta/2}/2\right)^2 + 1}. \end{aligned} \tag{59}$$

In all of these solutions the parameters c and p (see equations (A26) and (A35)) are taken such that the solitons and cuspons have an extremum at the point $\eta = 0$:

$$u_{sol}|_{\eta=0} = \frac{4}{3a^2} \qquad u_{cusp}|_{\eta=0} = -\frac{4}{3a^2}.$$

Note that the families of solutions represented by $\gamma = \frac{1}{4}$ and $\frac{3}{4}$ (by $\gamma = -\frac{1}{2}$ and $\frac{3}{2}$) involve the unbounded cuspon with a positive minimum (a soliton with a negative maximum) at the point $\eta = 0$. We will not write down these solutions here.

A thorough analysis should involve the numerical investigation of equation (35) for various values of the constants γ and will be done elsewhere.

3.3. Multi-cuspon solutions

Here we give some features of the solutions of the CH which are related to the multi-soliton solution of the deformed sinh–Gordon equation (34). As is shown in the appendix, this solution is related to the kernel R_0 of the form (A34). Our consideration is based on the fact that each soliton of the multi-soliton solution of the deformed sinh–Gordon equation has its own velocity. So on a large time scale all solitons will be separated from one another and will propagate with negligible interaction, and one can consider the transformation (14), (25) and (27) (or (A26), (A32) and (A33)) between equations (3) and (34) separately for each soliton. This can be done due to the fact that both single solitons of equation (34) and bounded solitary-wave solutions of the CH tend to zero at infinity. This can be understood on the basis of the analysis of equations (A26), (A32) and (A33) which can be done without obstacles. Unfortunately, we cannot give an explicit formula for the multi-solitary-wave solution of CH because equation (A26) becomes too complicated in this case. Even in the case of the simplest two-solitary wave solution these equations leads to a polynomial equation of fifth degree.

According to the previous paragraph the multi-soliton solution of equation (34) is mapped by equations (A32) and (A33) into the family of solutions of CH which involves either a pure multi-cuspon solution or pure multi-soliton solutions or mixed multi-cuspon/multi-soliton solutions where each single cuspon has negative amplitude while each soliton has a positive one.

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Appendix. The $\bar{\partial}$ problems for the CH and deformed sinh–Gordon equations

In this appendix we will obtain the relations given in section 2 by using the $\bar{\partial}$-problem.

The $\bar{\partial}$ -problems for equations (3) and (34) have been introduced in [10, 12]. First of all, we recall some important facts obtained in [12]. We then show that the $\bar{\partial}$-problem for CH immediately follows from these results and we will demonstrate that equation (4) can also be treated in a similar way.

The deformations of the sinh–Gordon equation (34) and (4) are integrated with the help of the integral equation (non-local $\bar{\partial}$-problem),

$$\varphi(\lambda; X, T) = \eta(\lambda) + \frac{1}{2\pi \mathrm{i}} \int \frac{\mathrm{d}\nu \wedge \mathrm{d}\bar{\nu}}{\nu - \lambda} \int \varphi(\mu; X, T) R(\mu, \nu; X, T)\, \mathrm{d}\mu \wedge \mathrm{d}\bar{\mu} \tag{A1}$$

where the kernel R is of the form

$$R(\mu, \lambda; x, t) = R_0(\mu, \lambda)\, \mathrm{e}^{K(\mu;X,T) - K(\lambda;X,T)} \qquad K(\lambda; X, T) = \lambda X + \frac{\gamma T}{\lambda} \tag{A2}$$

with a an arbitrary real point on the complex plane, $a \neq 0$ (the same a is used in equation (3)). The normalization function η should be equal to 1 or $1/(\lambda - a)$ in order to obtain equations (34) or (4), respectively (recall that it was shown that these equations are related by the Miura-type transformation (31) and (33)). In addition, the following reduction is imposed on R:

$$R(\mu,\lambda)(\Omega(\mu)-\Omega(\lambda))=0 \qquad \Omega(\lambda)=\frac{1}{\lambda(\lambda-a)}. \tag{A3}$$

The last statement means that the kernel R_0 is of the form

$$R_0(\nu,\mu)=r_0(\nu,\mu)(\delta(\mu-\nu)+\delta(\mu+\nu-a)) \tag{A4}$$

where δ is the Dirac delta function ($\int\delta(\lambda)\,\mathrm{d}^2\lambda=-2\mathrm{i},\ \mathrm{i}^2=-1$). The only requirement on the function r_0 is that equation (A1) is uniquely solvable.

We remind some points of [12] where equation (34) had been derived. As was mentioned above, this equation is related to the solution $\varphi=\psi$ of equation (A1), normalized with $\eta=1$. The solution ξ is expressed through the residue of the function $\psi(\lambda)$ at infinity

$$\psi\to 1+\frac{\psi_1}{\lambda}+\cdots \qquad \text{as} \quad \lambda\to\infty$$

by the formula

$$\psi_{1T}+1=\mathrm{e}^{\xi}. \tag{A5}$$

In terms of the function ψ the potentials A and v of the linear system (28) and (29) are expressed by the following relations:

$$A=\frac{\psi_X(0)}{\psi(0)} \qquad \psi(0)=\psi(\lambda)|_{\lambda=0} \qquad v=\psi_{1T}+1 \tag{A6}$$

(we are using the designations of [12]).

We are more closely interested in the deformation (4) which results from the solution of the $\bar\partial$-problem $\varphi=\psi_a$ with the normalization $\eta=1/(\lambda-a)$, because the dressing procedure for CH is based on it. We write down this $\bar\partial$-equation because it will be important in what follows:

$$\psi_a(\lambda;X,T)=\frac{1}{\lambda-a}+\frac{1}{2\pi\mathrm{i}}\int\frac{\mathrm{d}\nu\wedge\mathrm{d}\bar\nu}{\nu-\lambda}\int\psi_a(\mu;X,T)R(\mu,\nu;X,T)\,\mathrm{d}\mu\wedge\mathrm{d}\bar\mu. \tag{A7}$$

We also give two asymptotics of the solution ψ_a:

$$\psi_a\to\frac{\psi_{a1}}{\lambda}+\cdots \qquad \text{as} \quad \lambda\to\infty. \tag{A8}$$

$$\psi_a\to\frac{1}{\lambda-a}+\psi_{a1}(a)+\cdots \qquad \text{as} \quad \lambda\to a. \tag{A9}$$

Now we can derive the second deformation of the sinh–Gordon equation (4). The detailed $\bar\partial$-approach for the construction of the integrable nonlinear partial differential equations (PDEs) is given in [14, 15]. Here we give two general points on which the succeeding consideration is based.

(a) If the function φ is the solution of equation (A1) with given kernel R of the form (A2) and (A3) and some normalization η, then the function

$$M\varphi\equiv\sum_j u_j(X,T)\,\Omega^{l_j}D_X^{m_j}D_T^{n_j}\varphi \qquad D_X=\partial_X+\lambda \qquad D_T=\partial_T+\frac{1}{\lambda} \tag{A10}$$

(where l_j, m_j, n_j are integers, and u is an arbitrary function of its arguments) is the solution of equation (A1) with the same kernel R and some different normalization. If one has two different solutions φ_1, φ_2 of equation (A1) with the same kernel R and different normalization, then the function

$$M_i\varphi_1 + M_j\varphi_2 \qquad i, j \in \mathbb{Z} \tag{A11}$$

is the solution of the same equation with a new normalization (here operators M_i, M_j are both of the form (A10)).

(b) There exist operators $\tilde{M}_k$ ($k = 1, 2, \ldots$) of the form (A10), such that the functions $\tilde{M}_k\varphi$ and/or $\tilde{M}_i\varphi_1 + \tilde{M}_j\varphi_2$ are solutions of equation (A1) with zero normalization. Then, in view of the uniqueness of the solution of the integral equation (A1), one has

$$\tilde{M}_k\varphi = 0 \qquad \text{and/or} \qquad \tilde{M}_i\varphi_1 + \tilde{M}_j\varphi_2 = 0 \tag{A12}$$

for some i, j, k.

We now have two different ways to construct the nonlinear PDEs:

1. due to the structure of the operator $\tilde{M}_k$, equations (A12) are actually an overdetermined system of linear equations on the functions φ, φ_1, φ_2, with variable potentials. Its compatibility condition produces the nonlinear system of equations for these potentials;
2. the nonlinear PDE follows immediately from the non-trivial terms of the expansions of equations (A12) in powers of the small parameters ε_k, $k = 0, 1, 2$,

$$\varepsilon_0 = (1/\lambda)|_{\lambda\to\infty} \qquad \varepsilon_1 = \lambda|_{\lambda\to 0} \qquad \varepsilon_2 = (\lambda - a)|_{\lambda\to a} \tag{A13}$$

where a is the singularity of the operator Ω (A3).

We will choose the second way.

In the situation under consideration $\varphi \equiv \psi_a$, and one can construct the following overdetermined linear system of differential equations:

$$M_1\psi_a \equiv D_{XX}\psi_a - \frac{1}{\Omega}\psi_a - \bar{U}_2 D_X\psi_a + U_3\psi_a = 0 \tag{A14}$$

$$M_2\psi_a \equiv D_T\psi_a + V\Omega D_X\psi_a - aV\Omega\psi_a + W\psi_a = 0 \tag{A15}$$

where

$$V^{-1} = -\frac{1}{\psi_a(0)}\left(\psi_a(0) - \frac{\psi_{aX}(0)}{a}\right)$$

$$W = -\frac{1}{a}(1 + V\psi_{a1X}(a) + V)$$

$$\bar{U}_2 = \frac{1}{\psi_{a1}}(2\psi_{a1X} + a\psi_{a1})$$

$$U_3 = a\bar{U}_2 - a^2.$$

The examination of the first non-trivial terms in the expansions of equation (A14) in powers of the small parameters ε_1, ε_2 and equation (A15) in powers of the parameter ε_0, results in the following nonlinear equation on the function χ:

$$\begin{aligned} &\tfrac{1}{2}\chi_T - a\partial_X^{-1}(\mathrm{e}^{\chi})\partial_X^{-1}(\mathrm{e}^{-\chi}) - c_1\partial_X^{-1}\mathrm{e}^{\chi} - c_2\partial_X^{-1}\mathrm{e}^{-\chi} = \frac{1 - c_1c_2}{a} \\ &\partial_X(\chi) = \bar{U}_2 - 2a = 2\frac{\psi_{a1X}}{\psi_{a1}} - a \end{aligned} \tag{A16}$$

(compare with (4)). Of course, the solutions ψ and ψ_a are not independent, but related by the differential equation (see (A11) and (A12)):

$$\psi_{aX} + \lambda\psi_a = \psi_{a1}\psi + a\psi_a \tag{A17}$$

whose expansion in terms of the parameter ε_0 leads to the Miura-type transformations (31) and (33) (with $\chi \to -\chi$, because of the symmetry of the system (5) and (6) with respect to the change $U_2 \to -U_2$, $x \to -x$, $V \to -V$) between solutions of the nonlinear equations (34) and (4).

Now we are going over to the $\bar{\partial}$-problem for CH (3). Here we give a slightly different formulation of this problem in comparison with that given in [10]. This reformulation is more suitable for our consideration and does not affect the final results.

To begin with, let us go to the parameters (x, t) through equation (14), where Φ is an arbitrary function of its arguments. We write down the overdetermined linear system for the function ψ_a which immediately follows from the $\bar{\partial}$-problem (A2), (A3) and (A7) which can also be obtained from the system (A14) and (A15) by means of the transformation (14),

$$D_{xx}\psi_a - \frac{\tilde{U}_1}{\Omega}\psi_a - \tilde{\tilde{U}}_2 D_x\psi_a + \tilde{U}_3\psi_a = 0 \tag{A18}$$

$$D_t\psi_a - uD_x\psi_a + \tilde{V}\Omega D_x\psi_a - a\tilde{V}\Phi_x\Omega\psi_a + \tilde{W}\psi_a = 0 \tag{A19}$$

where

$$u = \frac{\Phi_t}{\Phi_x} \tag{A20}$$

$$\tilde{V}^{-1} = -\frac{1}{\psi_a(0)}\left(\Phi_x\psi_a(0) - \frac{\psi_{a\,x}(0)}{a}\right) \tag{A21}$$

$$\tilde{W} = -\frac{1}{a}(1 + \tilde{V}\psi_{a1_x}(a) + \tilde{V}\Phi_x) \tag{A22}$$

$$\tilde{U}_1 = \Phi_x^2 \tag{A23}$$

$$\tilde{\tilde{U}}_2 = \frac{1}{\Phi_x\psi_{a1}}(\Phi_{xx}\psi_{a1} + 2\Phi_x\psi_{a1_x} + a\tilde{U}_1\psi_{a1}) \tag{A24}$$

$$\tilde{U}_3 = a\tilde{U}_2\Phi_x - a^2\tilde{U}_1 - a\Phi_{xx}. \tag{A25}$$

This overdetermined system produces the general nonlinear system of PDEs obtained from equations (A14) and (A15) through the transformation (14). The arbitrary function Φ is there. We need to impose a restriction on this system in the form (22) ($\tilde{V} = 1$) to give the CH. This restriction is equivalent to the following algebraic equation for the function Φ:

$$\psi_a(0) = c(t)\exp(a(\Phi + x)) \tag{A26}$$

(c is an arbitrary function of t, which will be clarified later). Recall that the left-hand side of this equation depends explicitly on the function Φ due to the transformation (14).

It will become clear that the potential u of equation (A19) is the solution of CH (3). To show this, we have to express all other potentials of the system (A18) and (A19) in terms of u. To do so, one needs to consider the non-trivial terms of the expansions of equations (A18) and (A19) in powers of the parameters ε_k, $k = 0, 1, 2$ (A13) and take into account equation (A26). After some rather simple transformations one obtains

$$\tilde{U}_2 = a(1 + 2\Phi_x) \tag{A27}$$

$$\tilde{W} = \frac{1}{2}\left(u_x + au - \frac{b_t(t)}{b(t)}\right) \tag{A28}$$

$$\tilde{U}_1 = -\frac{u_{xx}}{2} + a^2\frac{u}{2} + 1 + \frac{b_t}{b}. \tag{A29}$$

The function $b(t)$ which appears in these formulae is related to the function $c(t)$. In fact, let us take the expansion of equation (A17) in powers of the parameter ε_1. In view of (A24) and

(A26) one has

$$\psi(0) = \frac{c(t)}{\sqrt{b(t)}}\frac{1}{\sqrt{\Phi_x}}\exp\left(\tfrac{1}{2}a(\Phi + x)\right). \tag{A30}$$

Note that the left-hand side of this equation does not depend on any arbitrary function of t. This means that $c/\sqrt{b} = 1$, in the right-hand side of this equation. Moreover, to obtain the CH equation with constant potentials one needs to take the function $b(t)$ such that the function (b_t/b) would be a constant. We take $b =$ constant for simplicity. Finally, for the function Ψ_a

$$\Psi_a = \psi \mathrm{e}^{K-a\Phi} \tag{A31}$$

one obtains the system (23) and (24).

After this discussion, we obtain the CH equation (3), which can be obtained either as the compatibility condition of the system (A18) and (A19) or from consideration of the non-trivial terms in the expansions of equations (A18) and (A19) in powers of the parameters ε_k, $k = 0, 1, 2$ (A13). Some of these non-trivial terms lead to equations (A27)–(A29). Other ones allow us to construct equation (3).

The connection between the solutions of the deformed sinh–Gordon equations (4), (34) and CH (3) is obtained from equations (A17), (A24) and (A26). Finally, it can be written as (compare with (27))

$$\psi_{a1}^2\Phi_x = \mathrm{e}^{(x+\Phi)a+b(t)} \tag{A32}$$

($b(t) =$ constant from above) in view of the definitions (A16) and (A20) and of the Miura transformation (31),

$$\partial_X(\chi) = 2\partial_X \ln(\psi_{a1}) - a \qquad \ln_X(\xi) = \tfrac{1}{4}\left(\chi_X^2 - \chi_{XX}\right)_T \qquad u = \frac{\Phi_t}{\Phi_x}. \tag{A33}$$

Note that equation (A32) has, in general, more than one different solution. This means that any solution of the deformed sinh–Gordon equation is related to a family of solutions of the CH equation. The number of solutions in this family depends on the solution of the deformed sinh–Gordon equation, which produces this family. The problem of constructing real solutions of CH should be solved for each family independently. One can say at least that real solutions of equation (A32) give rise to real solutions of CH (3).

Now we turn to the soliton solutions of equation (34). The N-soliton solution is represented by equations (A1)–(A3) with $\eta = 1$ and the kernel R_0 of the form

$$R_0(\nu, \mu) = \delta(\nu - a + \mu)\sum_{k=1}^{N} p_k\delta(\nu - b_k) \tag{A34}$$

with p_k and b_k arbitrary real constants. We consider the one-soliton solution, when equation (A34) has only one term ($N = 1$, $b_1 = b$, $p_1 = p$) and rewrite this equation in the form

$$R_0(\nu, \mu) = -\tfrac{1}{2}\pi\mathrm{i}\delta(\mu - a + \nu)\,\delta(\nu - b)p. \tag{A35}$$

To construct the solution of CH (3), we take equation (A26) and the $\bar{\partial}$-problem given by (A7) (instead of (A1)) with the same kernel R_0. Note that the $\bar{\partial}$-technique allows us to construct the solutions of CH related to the solutions of equation (34) without solving the differential equation (A32). We should simply use the algebraic formula (A26) to define the function Φ and formula (A20) to find the solution of CH.

Equation (A7) with the kernel (A2), (A3) and (A35) can be easily integrated resulting in

$$\psi_a(\lambda) = \frac{1}{\lambda - a} + \frac{p}{a - b - \lambda}\psi(b)\chi \tag{A36}$$

where we have introduced the function χ, which is related to Φ by the formula

$$\chi = \exp\left[(2b - a)\Phi + \frac{2b - a}{b(b - a)}t\right]. \tag{A37}$$

The function $\psi(b)$ can be defined from equation (A36) by substituting $\lambda = b$ in it,

$$\psi(b) = \frac{1}{(b - a)}\frac{1}{(1 + (p/(2b - a))\chi)}. \tag{A38}$$

Then equation (A26) for the function χ is reduced to the following one:

$$ac(\gamma - 1)^2(a - 2a\gamma - p\chi)\chi^{1/(2\gamma-1)}\,\mathrm{e}^{a\eta} - p\gamma^2\chi - a(2\gamma - 1)(\gamma - 1)^2 = 0 \tag{A39}$$

where $\gamma = b/a$, $\beta = 1/[a^2\gamma(1 - \gamma)]$, $\eta = x + \beta t$.

The N-soliton solution of equation (34) can be considered without difficulties. In this case equation (A26) is also an algebraic one of degree P, $P > 4$ and has to be solved numerically.

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